Discretization of frequencies in delay coupled oscillators

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We study the dynamics of two mutually coupled chaotic oscillators with a time delayed coupling. Due to the delay, the allowed frequencies of the oscillators are shown to be discretized. The phenomenon is observed in the case when the delay is much larger than the characteristic period of the solitary uncoupled oscillator.

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I. INTRODUCTION

The goal of this paper is to consider the influence of delay on the dynamics of coupled oscillators. In particular, we study the phenomenon of discretization of frequencies, which arises due to a delay in the coupling. The effect was recently observed in [1] for a system of coupled lasers. The authors there argued that it should a general property of coupled oscillators as well. We show that this effect persists even in the case when the oscillators are chaotic. We also provide a more detailed description of the phenomenon using coupled Kuramoto systems.

Dynamical properties of instantaneously coupled oscillators have been the subject of extensive research during the last decades [2,3]. Many new collective phenomena have been discovered and understood such as complete synchronization [4], generalized [5], phase [6], and lag [7] synchronization, clustering [8], etc. At the same time, the study of coupled systems appears to be important for many practical applications such as laser dynamics [9,10], biology [11], neurophysiology [12], chemistry [13], and others.

It is evident that a delay in the coupling is common, since coupled subsystems are usually located discretely in space. There is also evidence that the delay can change the dynamics significantly [14]. As soon as the delay becomes comparable with the period of oscillations of the solitary system, correct modeling should take it into account. The resulting systems of coupled oscillators with delay possess new features and exhibit new phenomena, e.g., anticipated synchronization [15]. Moreover, such models are more complicated objects to study [16,17] and determining the properties of delay coupled systems is still a challenging problem.

Section II presents the main results of the paper using the model of coupled Rössler oscillators. More deep analysis is done in Sec. III, where coupled Kuramoto systems with delay are considered. In particular, we show that a multistability of the allowed frequency levels can occur via a sequence of tangent bifurcations.

II. TWO DELAY COUPLED RÖSSLER OSCILLATORS

In this paper we consider the well studied paradigm of Rössler oscillators, which are bidirectionally coupled,

$$x'(t) = f_{\omega_1}(x(t)) + ky(t-\tau),$$

$$y'(t) = f_{\omega_2}(y(t)) + kx(t-\tau), \qquad (1)$$

where $x, y \in R^3$ are vectors, $f_\omega(x) = (-\omega x_2 - x_3, \omega x_1 + a x_2, b + x_3(x_1 - c))^T$, $\tau > 0$ is the delay time of the coupling, and k is the coupling strength. Note that a similar coupling configuration appears in a system describing two optically coupled semiconductor lasers [1,10,15].

In order to distinguish between periodic and chaotic cases, we use c as the control parameter, which determines the regularity of the solitary system. First, we fix $a=0.15, b=0.4$, and the base frequency $\omega=1$. It is well known that changing c one can observe a period doubling route to chaos in a single Rössler system. Hence, in what follows, $c=4$ will correspond to the periodic case and $c=8.5$ to the chaotic one. For completeness, we will also consider the intemediate case $c=7$, when the uncoupled oscillator with $\omega=1.02$ is periodic and the other with $\omega=0.98$ is chaotic.

For the considered parameter values, one can introduce phases of the oscillators in a simple geometric manner [2]. Formally, the phases can be defined as $\varphi_1(t) = \arctan[x_2(t)/x_1(t)] + \pi n_1(t)$, $\varphi_2(t) = \arctan[y_2(t)/y_1(t)] + \pi n_2(t)$, where $n_i(t)$ are integer valued functions chosen in such a way that $\varphi_i(t)$ are continuous. A practical way to compute φ_i is to introduce polar coordinates in Eq. (1). The mean observed frequencies of the oscillators are $\Omega_i = \lim_{t\to\infty} \varphi_i(t)/t, i=1, 2$. Synchronization properties of instantaneously ($\tau=0$) mutually coupled systems are well studied for both periodic and chaotic systems [2]. In both cases one observes frequency synchronization regions, which correspond to the case $\Omega_1 = \Omega_2$. These regions have the form of cones in the parameter space detuning–coupling strength, i.e., $\Delta\omega = \omega_1 - \omega_2$ and k. Figures 1(a)–1(c) and 2(a) illustrate the dependence of Ω_i and $\Omega_2 - \Omega_1$ on $\Delta\omega$ for fixed $k=0.005$ and $\tau=0$. We observe the "classical" synchronization plateau and a smooth dependence of the frequencies on the control parameter.[1] In Fig. 3 (left panel) we compute the corresponding Lyapunov expo-

[1]In fact, the dependence of $\Omega_i(\Delta\omega)$ has the devil's staircase structure. Nevertheless, it is numerically indistinguishable from a continuous function, provided there are no main resonances. The chaotic behavior smears the staircase structure as well.

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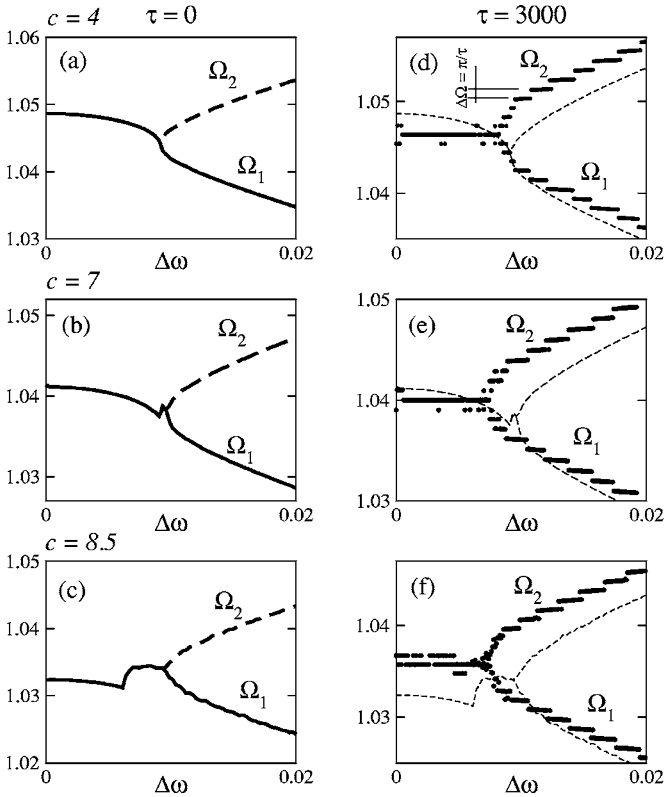

FIG. 1. (a)–(c) Mean frequencies Ω_1 and Ω_2 for instantaneously coupled systems $k=0.005, \tau=0$. (d)–(f) The same for delay coupled systems with $\tau=3000$ and $k=0.005$. Different rows correspond to different values of c, as indicated in the figure. $c=8.5$ and 7 stand for the chaotic and $c=4$ for the regular case.

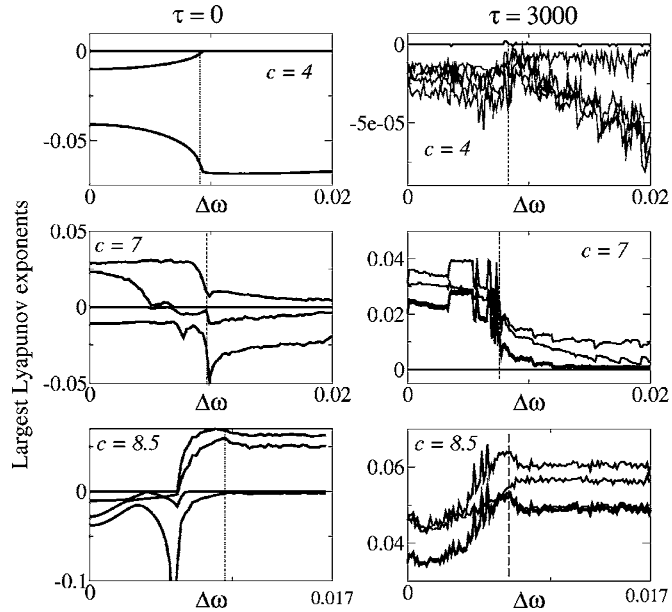

FIG. 3. Largest Lyapunov exponents as functions of $\Delta\omega$. The left panel corresponds to the instantaneous coupling and the right one to the delayed case with $\tau=3000$. The dashed vertical lines mark the parameter value at which the phase synchronization transition happens.

nents, which indicate that the dynamics for $c=4$ remains regular for all values of $\Delta\omega$ while for $c=7$ it is chaotic. For $c=8.5$ it is also chaotic except for a region of small detuning within the synchronization domain.

The effect of the delay that we would like to report here is illustrated in Figs. 1(d)–1(f). We plot there the same quantities as in Figs. 1(a)–1(c) but for delay coupled oscillators. Instead of the continuous behavior of the frequencies with changing $\Delta\omega$, we observe a "quantization" effect when some preferable values of frequencies appear which destroy the previously continuous dependence on the parameters. Ω_i undergo jumps of magnitude π/τ with varying $\Delta\omega$. As illustrated in Fig. 1(d), the allowed values of the frequencies and the jumps are closely related to the round-trip frequency $\omega_f = \pi/\tau$. From this point of view, one can interpret this phenomenon as resonances to multiples of the round-trip frequency. Figure 3 (right panel) shows the largest Lyapunov exponents for the case with delay. One can note that the case $c=4$ still corresponds to a regular dynamics and $c=7$ and 8.5 to chaotic. Hence, the observed phenomenon takes place for chaotic oscillators as well. Note that in the chaotic case there are many (at least more than ten) positive Lyapunov exponents which behave similarly to each other. This phenomenon is in agreement with recent results on delay systems with large delay [17].

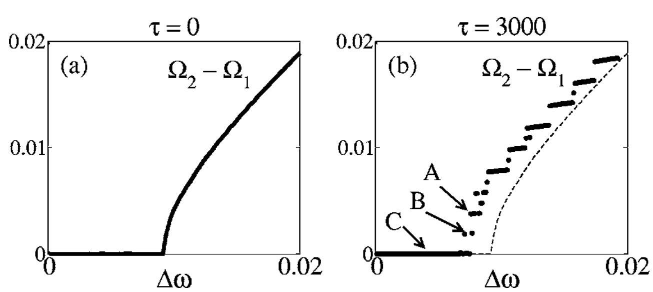

FIG. 2. Dependence of $\Omega_2-\Omega_1$ on the detuning. $k=0.005$, $c=4$. $\tau=$ (a) 0; (b) 3000.

In our simulations we choose $\tau=3000$.[2] We were not able to observe the discretization phenomenon for small values of τ, which are comparable with the characteristic period of the Rössler oscillator $\omega_R \approx 1$, i.e., we have $\pi/\tau < \Delta\omega \ll \omega_R$. Considering the coupled Kuramoto system in the next section, we will argue for the large delay.

Figure 4 shows the evolution of the phase difference $\varphi_1 - \varphi_2$ for $c=4$ and $c=8.5$. In particular, in Fig. 4(a) the orbits A, B, and C have been computed for three different values of detuning, which correspond to three minimal allowed frequencies $\Omega_2-\Omega_1$; cf. also the points A, B, and C in Fig. 2(b). It is interesting to note that phase slips in both nonsynchronous cases A and B occur with the same rate, but in the case A these slips have the magnitude 2π, while in the case B the

[2]In order to compute the mean frequencies, we have used the modified Runge-Kutta fourth-order method with fixed step $h=0.1$. For each parameter value, the same initial conditions have been used, $x_i(\theta)=y_i(\theta)=1.0$, where $\theta \in (-\tau, 0)$. The averaging has been performed over the interval $T_{av}=4\times 10^5 \approx 133\tau$ after the transient $T_t=2\times 10^5 \approx 67\tau$. Lyapunov exponents have been computed using the method described in [18].

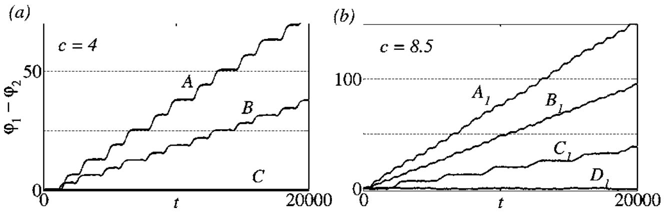

FIG. 4. Evolution of phase differences of delay coupled oscillators. Parameter values for (a) $c=4$; orbit C, $\omega_1=0.998$, $\omega_2=1.002$ (phase synchronized case); B, $\omega_1=0.9962$, $\omega_2=1.0038$ (minimal allowed frequency difference); C, $\omega_1=0.9961$, $\omega_2=1.0039$. The corresponding points are also indicated in Fig. 2(b). Parameter values for (b) A_1, $\omega_1=0.9955$, $\omega_2=1.0045$; B_1, $\omega_1=0.9954$, $\omega_2=1.0045$; C_1, $\omega_1=0.9963$, $\omega_2=1.0037$; D_1, $\omega_1=0.9975$, $\omega_2=1.0025$.

magnitude of the slips is π. Figure 5 illustrates this in more detail.

Our observations suggest that in the case with delay, transition to the phase synchronization differs from those that occur in the instantaneous case [2]. In particular, the scaling properties of the intervals between phase slips can be different. We will report the scaling results elsewhere. Our analysis of the phase slips indicates that the mutual synchronization of two oscillators can be superimposed by a synchronization of the phase slips to integer fractions of the round-trip time.

III. DELAY COUPLED PHASE OSCILLATORS

In order to obtain an additional insight into the appearance of the discretized frequency levels, let us consider the Kuramoto model with delay,

$$\psi_1'(t) = \omega_1 - k\sin[\psi_1(t) - \psi_2(t-\tau)],$$

$$\psi_2'(t) = \omega_2 - k\sin[\psi_2(t) - \psi_1(t-\tau)]. \qquad (2)$$

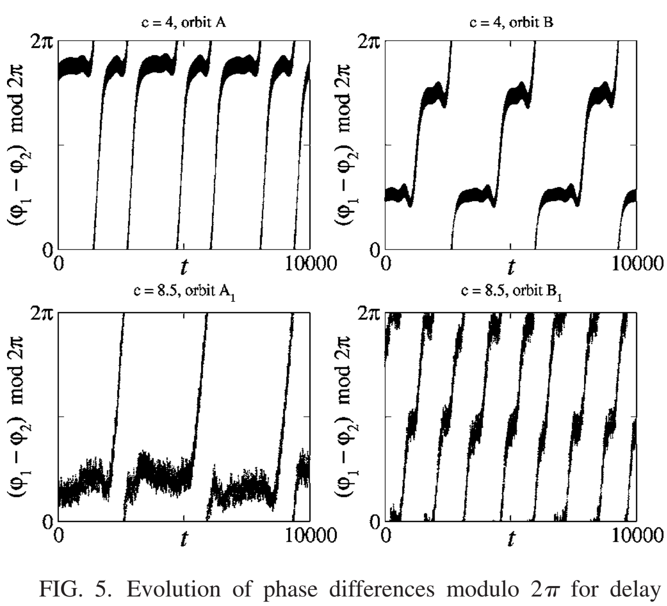

FIG. 5. Evolution of phase differences modulo 2π for delay coupled oscillators.

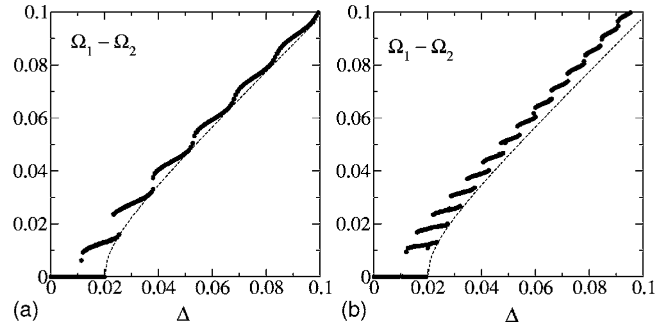

FIG. 6. Frequencies of system (2) for different values of τ versus Δ. (a) the case with $\tau=400$; (b) $\tau=1000$. In both figures $k=0.01$.

In order to show how the dynamics of the system changes with increasing delay, let us extend the analysis of [1]. For simplicity, we assume that $\omega_1=-\omega_2=\Delta/2$. The numerical analysis of system (2) in Fig. 6 reveals the following qualitative features of the observed phenomenon: as the delay increses, the continuous curve of stable nonsynchronous periodic solutions change its curvature and, at some moments, S-shaped parts appear, which apparently indicates the appearance of tangent bifurcations. As a result of such bifurcations, multistability develops with increasing delay.

Within the locking region, system (2) is known [20] to exhibit a series of synchronized solutions of the form $\psi_{1,2}=\Omega t \pm \alpha/2$, where Ω and α are constants. The stability and existence of such solutions have been studied in [20]. Unfortunately, we cannot apply their results here in order to support our calculations, since the described phenomenon goes beyond the simple "constant frequency" solutions. Instead, we would like to characterize a small neigborhood of a stationary solution using the analytical technique developed in [17]. Particularly, we are going to show that in the neighborhood of each of the constant frequency solutions, the oscillations can occur only with the frequencies, which are commensurable with π/τ (regardless of whether they are damped or not). Assuming that $\psi^0_{1,2}=\Omega t \pm \alpha/2$ is a solution of (2) with some given Ω and α, the linearized system, which determines the stability of $\psi^0_{1,2}$, reads

$$\xi_1'(t) = -k\cos(\alpha+\Omega\tau)\xi_1(t) + k\cos(\alpha+\Omega\tau)\xi_2(t-\tau),$$

$$\xi_2'(t) = -k\cos(\alpha-\Omega\tau)\xi_2(t) + k\cos(\alpha-\Omega\tau)\xi_1(t-\tau). \qquad (3)$$

As follows from [17], the possible imaginary parts of the critical eigenvalues are approaching asymptotically the values $\omega_H=\arg(\mu)/\tau + 2\pi n/\tau$ as τ becomes large. Here $\arg(\cdot)$ denotes the argument of a complex number and μ is a zero of the following equation:

$$\det\begin{bmatrix} k\mu\cos(\alpha+\Omega\tau) & -k\cos(\alpha+\Omega\tau) \\ -k\cos(\alpha-\Omega\tau) & k\mu\cos(\alpha-\Omega\tau)\end{bmatrix} = 0. \qquad (4)$$

From Eq. (4) we have $\mu=\pm 1$. Therefore, the only possible imaginary parts for the eigenvalues are $\omega_H = \pi/\tau + \pi n/\tau$, which coinside with the detected numerically available frequencies in Figs. 1(d)–1(f) and 3(b) for Rössler systems. This is analytical evidence that there are preferred frequencies in the model. A key condition for the application of the asymptotic analysis from [17] is the assumption that the delay is large $\tau \gg 1/(\omega_1 - \omega_2)$. We should admit also that this argumentation works for solutions in the vicinity of constant frequency solutions.

IV. CONNECTION BETWEEN COUPLED OSCILLATORS AND SYSTEMS WITH DELAYED FEEDBACK

In the following we would like to present some additional arguments showing that the described phenomenon is generic. Let us introduce an artificial parameter k_1 such that system (1) admits the form

$$x'(t) = f_{\omega_1}(x(t)) + ky(t) + k_1[y(t-\tau) - y(t)],$$

$$y'(t) = f_{\omega_2}(y(t)) + kx(t) + k_1[x(t-\tau) - x(t)]. \qquad (5)$$

System (5) coincides with (1) if $k = k_1$ while at $k_1 = 0$ it has instantaneous coupling. Therefore, increasing the parameter k_1 from 0 to k, the case with instantaneous coupling is transformed to the delayed one. In a short form, Eq. (5) can be written as

$$z' = F(z) + Kz(t) + K_1[z(t-\tau) - z(t)], \qquad (6)$$

where $z = (x, y)^T$, $K = \begin{pmatrix} 0 & kI_3 \\ kI_3 & 0 \end{pmatrix}$, $K_1 = \begin{pmatrix} 0 & k_1 I_3 \\ k_1 I_3 & 0 \end{pmatrix}$, and I_3 is a 3×3 unit matrix. Our main observation is that the system (6) can be considered as the instantaneously coupled system $z' = F(z) + Kz(t)$ under the action of the feedback term $K_1[z(t-\tau) - z(t)]$. As follows from [19], this term, under some conditions, enhances the spectral properties of the solutions, e.g., it stabilizes periodic solutions with periods close to fractions of τ, for which the feedback term vanishes. Roughly speaking, such a feedback induces a filtering of frequencies that are close to multiples of $2\pi/\tau$. Following this idea, one may consider (6), and hence (1) as well, as an instantaneously coupled system, which undergoes the influence of the delayed feedback. As a result, the frequencies of the instantaneous system (see Fig. 1) are "filtered" through the delayed feedback term and one observes an enhancing of those frequencies, that are multiples of $2\pi/\tau$.

V. DISCUSSION

To summarize, we report the phenomenon of frequency discretization in systems of chaotic oscillators. A large delay is shown to be essential for its appearance. It is still a challenging problem to show whether this phenomenon can be detected in systems with nonisochronous attractor. The main difficulty, which we have to face in this case, is the problem of phase definition in such systems. As a rule, the notion of frequency in such systems is ambiguous and can be determined up to a limited precision (see, e.g. the method from [21]). On the other hand, the characteristic frequency splitting that we have observed is of the magnitude π/τ. Thus, for large τ, the frequencies have to be determined with high precision in order to resolve this phenomenon.

Note that the above mentioned conditions are very natural for optical systems and, in particular, for semiconductor lasers with optical feedback [1,22]. Therefore it is not a surprise that its first occurrence comes from this field [1]. Nevertheless, our arguments show that this generic phenomenon can occur in systems of completely different nature, e.g., biological or mechanical systems, as soon as the internal time scales of oscillators become smaller than the time scale of the interaction between them.

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[1] H.-J. Wünsche, S. Bauer, J. Kreissl, O. Ushakov, N. Korneyev, F. Henneberger, E. Wille, H. Erzgräber, M. Peil, W. Elsäßer, and I. Fischer, Phys. Rev. Lett. **94**, 163901 (2005).

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, U.K., 2001).

[3] E. Ott, *Chaos in dynamical systems*, 2nd ed. (Cambridge University Press, Cambridge, U.K., 2002); E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic Synchronization: Application to Living Systems* (World Scientific, Singapore, 2002); C.-W. Wu, *Synchronization in Coupled Chaotic Circuits and Systems* (World Scientific, Singapore, 2002).

[4] H. Fujisaka *et al.*, Prog. Theor. Phys. **69**, 32 (1983); L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); L. Pecora *et al.*, Chaos **7**, 520 (1997); S. Yanchuk *et al.*, *ibid.* **13**, 388 (2003); S. Yanchuk *et al.*, Physica D **154**, 26 (2001).

[5] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).

[6] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[7] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).

[8] K. Kaneko, Physica D **41**, 137 (1990). S. Yanchuk, Y. Maistrenko, and E. Mosekilde, Math. Comput. Simul. **54**, 491 (2001).

[9] R. Roy and K. S. Thornburg, Phys. Rev. Lett. **72**, 2009 (1994).

[10] S. Yanchuk, K. R. Schneider, and L. Recke, Phys. Rev. E **69**, 056221 (2004).

[11] H. Bohr *et al.*, Parallel Comput. **12**, 113 (1989); R. C. Elson *et al.*, Phys. Rev. Lett. **81**, 5692 (1998); E. Mosekilde *et al.*, BioSystems **63**, 3 (2001).

[12] P. A. Tass, T. Fieseler, J. Dammers, K. Dolan, P. Morosan, M.

Majtanik, F. Boers, A. Muren, K. Zilles, and G. R. Fink, Phys. Rev. Lett. **90**, 088101 (2003).

[13] I. Z. Kiss, Y. Zhai, and J. L. Hudson, Phys. Rev. Lett. **88**, 238301 (2002).

[14] T. Heil, I. Fischer, W. Elsäßer, J. Mulet, and C. R. Mirasso, Phys. Rev. Lett. **86**, 795 (2001).

[15] H. U. Voss, Phys. Rev. E **61**, 5115 (2000); A. Locquet, C. Masoller, and C. R. Mirasso, *ibid.* **65**, 056205 (2002).

[16] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations* (Springer-Verlag, Berlin, 1993).

[17] S. Yanchuk, Math. Methods Appl. Sci. **28**, 363 (2005); S. Yanchuk and M. Wolfrum (unpublished).

[18] J. D. Farmer, Physica D **4**, 366 (1982).

[19] K. Pyragas, Phys. Lett. A **170**, 421 (1992).

[20] H. G. Schuster and P. Wagner, Prog. Theor. Phys. **81**, 939 (1989).

[21] M. G. Rosenblum, A. S. Pikovsky, J. Kurths, G. V. Osipov, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett. **89**, 264102 (2002),

[22] T. Sano Phys. Rev. A **50**, 2719 (1994),